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Exact results of a general spin-1 model on 2D decorated lattices

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Abstract. By the decoration-iteration transformation, we find a general relation between interaction parameters of a general spin-1 model on any lattice with unique coordination number and on its decorated lattice. We obtain three exact phase transition points of the general spin-1 model on two-dimensional decorated lattices in terms of the exact results of the model on corresponding two-dimensional regular lattices, including the triangle, square, kagomé and honeycomb lattices. We find the critical temperatures have no exact relation with the average coordination number and the type of coordination number for both the Ising model and the general spin-1 model.

1. Introduction

The spin-1 model has drawn much attention recently because it can be used to describe phase separation and superfluid ordering in ^{3}He–^{4}He mixtures [1] or phase separation and ferromagnetism in binary alloys [2], and presents a rich variety of critical and multicritical phenomena. It has been extensively studied by means of approximation methods such as the mean-field theory [1, 3–5], the Bethe approximation [6], correlated effective-field theory [11], renormalization group techniques [7–9], series expansion methods [10], Monte Carlo methods [12] and the cluster-variation method [13]. However, the exact solutions of the model are very limited. They have been obtained mainly on honeycomb lattices with the coordination number $\gamma_h = 3$ [15–17], and in some limit conditions, for example $|\Delta| \gg 1$ [8] or on certain ideal structures such as the Cayley tree [14]. Mi and Yang [18] have brought out a kind of universal non-one-to-one transformation between spin variables of two lattice models, specifically say the spin-1 model and the Ising model, which is applicable to all lattices and depends on the lattice structure only through the coordination number. They have therefore found some exact results of the spin-1 model on two-dimensional lattices with unique coordination number, including the triangle, square, kagomé and honeycomb lattices [18–20].

In this paper, we investigate the spin-1 model on two-dimensional decorated lattices. This sort of lattice with heterogeneous structure has two different coordination numbers. In this case, there exists some difficulty when we directly make use of the non-one-to-one transformation, but we find a general relation between interaction parameters of the model

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on any lattice with unique coordination number and on its decorated lattice by decoration-iteration transformation [21]. Then employing the exact results of the latter, we obtain those of the former.

In section 2, the decoration-iteration transformation is described and the exact critical points of the model for two-dimensional decorated lattices, in terms of the exact results of the model on regular two-dimensional lattices, are found. The results show that the critical temperatures have no exact relation with the average coordination number and the type of coordination number, for both the Ising model and the general spin-1 model. In section 3, we give a brief summary.

2. Decoration-iteration transformation and exact results for two-dimensional decorated lattices

The Hamiltonian of the general spin-1 model [4, 7, 8, 17, 18] on a regular two-dimensional lattice $\mathcal{L}$ such as the triangle, square, kagomé or honeycomb lattice is written as

$$-\beta\mathcal{H}_{\mathcal{L}} = J\sum_{\langle i,j\rangle} S_i S_j + \frac{L}{2}\sum_{\langle i,j\rangle} S_i S_j (S_i + S_j) + K\sum_{\langle i,j\rangle} S_i^2 S_j^2 + H\sum_{\langle i\rangle} S_i - \Delta\sum_{\langle i\rangle} S_i^2$$
$$S_i = 0, \pm 1 \tag{1}$$

where $\langle i, j\rangle$ indicates summation over the nearest-neighbour pairs of sites. J, L and K denote the reduced interaction parameters, H the external field and Δ the crystal field.

We write down the Hamiltonian of the general spin-1 model on a corresponding decorated lattice $\mathcal{L}_{(1)}$ as

$$-\beta\mathcal{H}_{\mathcal{L}_{(1)}} = J_1\sum_{\langle i,j\rangle} S_i S_j + \frac{L_1}{2}\sum_{\langle i,j\rangle} S_i S_j (S_i + S_j) + K_1\sum_{\langle i,j\rangle} S_i^2 S_j^2 + H_1\sum_{\langle i\rangle} S_i - \Delta_1\sum_{\langle i\rangle} S_i^2$$
$$S_i = 0, \pm 1 \tag{2}$$

where subscript 1 shows that the decorated lattice in study grows out of the initial lattice with one site decorated in each bond. As an example, we take the square lattice as the lattice $\mathcal{L}$ and its one-site decorated lattice as $\mathcal{L}_{(1)}$, which are shown in figure 1.

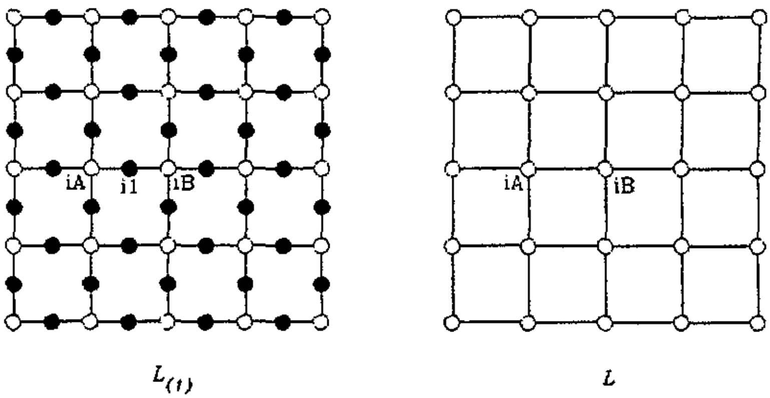

Figure 1. The square lattice $\mathcal{L}$ and its one-site decorated lattice $\mathcal{L}_{(1)}$. The open circle has coordination number four and the black circle two.

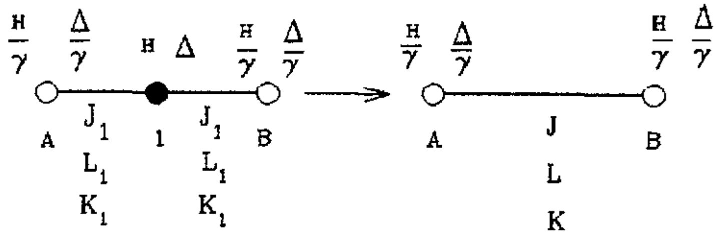

Figure 2. The basic decoration-iteration transformation of the general spin-1 model.

The partition function with the Hamiltonian $\mathcal{H}_{\mathcal{L}_{(1)}}$ (2) can be written as

$$
\begin{aligned}
Z_{\mathcal{L}_{(1)}} &= \sum_{\{S_i=0,\pm1\}} \exp\{-\beta\mathcal{H}_{\mathcal{L}_{(1)}}\{S_i\}\} \\
&= \sum_{\{S_i\}}^{\mathcal{L}} \prod_{\{b_i\}}^{\mathcal{L}} \sum_{\{S_{i1}\}} \exp\Bigg[J_1(S_{iA}S_{i1} + S_{i1}S_{iB}) \\
&\quad + \frac{L_1}{2}(S_{iA}^2 S_{i1} + S_{iA}S_{i1}^2 + S_{i1}^2 S_{iB} + S_{i1}S_{iB}^2) + K_1(S_{iA}^2 S_{i1}^2 + S_{i1}^2 S_{iB}^2) \\
&\quad + H_1\left(\frac{S_{iA}}{\gamma} + \frac{S_{iB}}{\gamma} + S_{i1}\right) - \Delta_1\left(\frac{S_{iA}^2}{\gamma} + \frac{S_{iB}^2}{\gamma} + S_{i1}^2\right)\Bigg] \\
&= \sum_{\{S_i\}}^{\mathcal{L}} B^{b_L} \prod_{\{b_i\}}^{\mathcal{L}} \exp\Bigg[J S_{iA}S_{iB} + \frac{L}{2}(S_{iA}^2 S_{iB} + S_{iA}S_{iB}^2) + K S_{iA}^2 S_{iB}^2 \\
&\quad + \frac{H}{\gamma}(S_{iA} + S_{iB}) - \frac{\Delta}{\gamma}(S_{iA}^2 + S_{iB}^2)\Bigg] \\
&= B^{b_l} Z_{\mathcal{L}}
\end{aligned} \tag{3}
$$

where $\sum_{\{S_i\}}^{\mathcal{L}}$ denotes summation over all sites on $\mathcal{L}$ and $\prod_{\{b_i\}}^{\mathcal{L}}$ multiplication over all bonds on $\mathcal{L}$. γ is the coordination number, B a constant, b_L the number of bonds of $\mathcal{L}$, and $Z_{\mathcal{L}}$ the partition function of the Hamiltonian (1).

It is easily seen from figure 2 that the basic decoration-iteration transformation between $\mathcal{L}$ and $\mathcal{L}_{(1)}$ is

$$
\begin{aligned}
&\sum_{\{S_1=0,\pm1\}} \exp\Bigg[J_1(S_A S_1 + S_1 S_B) + \frac{L_1}{2}(S_A^2 S_1 + S_A S_1^2 + S_1^2 S_B + S_1 S_B^2) \\
&\qquad + K_1(S_A^2 S_1^2 + S_1^2 S_B^2) + H_1\left(\frac{S_A}{\gamma} + \frac{S_B}{\gamma} + S_1\right) - \Delta_1\left(\frac{S_A^2}{\gamma} + \frac{S_B^2}{\gamma} + S_1^2\right)\Bigg] \\
&\quad = B \exp\Bigg[J S_A S_B + \frac{L}{2}(S_A^2 S_B + S_A S_B^2) + K S_A^2 S_B^2 \\
&\qquad + \frac{H}{\gamma}(S_A + S_B) - \frac{\Delta}{\gamma}(S_A^2 + S_B^2)\Bigg].
\end{aligned} \tag{4}
$$

Letting $S_A = 1$ and $S_B = 1$, $S_A = -1$ and $S_B = -1$, $S_A = 1$ and $S_B = 0$ (or $S_A = 0$ and $S_B = 1$), $S_A = -1$ and $S_B = 0$ (or $S_A = 0$ and $S_B = -1$), $S_A = 1$ and $S_B = -1$ (or

$S_A = -1$ and $S_B = 1$) and $S_A = 0$ and $S_B = 0$, respectively, we get a set of six equations:

$$\left\{\begin{array}{rcl} e^{2H_1-2\Delta_1}[e^{L_1+2K_1-\Delta_1}(e^{2J_1+L_1+H_1}+e^{-2J_1-L_1-H_1})+1]^{\gamma} &=& B^{\gamma}e^{\gamma J+\gamma L+\gamma K+2H-2\Delta} \\ e^{-2H_1-2\Delta_1}[e^{-L_1+2K_1-\Delta_1}(e^{-2J_1+L_1+H_1}+e^{2J_1-L_1-H_1})+1]^{\gamma} &=& B^{\gamma}e^{\gamma J-\gamma L+\gamma K-2H-2\Delta} \\ e^{H_1-\Delta_1}[e^{L_1/2+K_1-\Delta_1}(e^{J_1+L_1/2+H_1}+e^{-J_1-L_1/2-H_1})+1]^{\gamma} &=& B^{\gamma}e^{H-\Delta} \\ e^{-H_1-\Delta_1}[e^{-L_1/2+K_1-\Delta_1}(e^{-J_1+L_1/2+H_1}+e^{+J_1-L_1/2-H_1})+1]^{\gamma} &=& B^{\gamma}e^{-H-\Delta} \\ e^{-2\Delta_1}[e^{2K_1-\Delta_1}(e^{L_1+H_1}+e^{-L_1-H_1})+1]^{\gamma} &=& B^{\gamma}e^{-\gamma J+\gamma K-2\Delta} \\ [e^{\Delta_1}(e^{H_1}+e^{-H_1})+1]^{\gamma} &=& B^{\gamma}. \end{array}\right. \tag{5}$$

Table 1. Summaries of three exact critical points P_1, P_2 and P_3 for the general spin-1 model on a two-dimensional lattice $\mathcal{L}$. J_L^* is the critical point of the Ising model on the lattice $\mathcal{L}$ such as the triangle, square, kagomé and honeycomb lattices.

	J^*	L^*	K^*	H^*	Δ^*
P_1	J_L^*	$-2J_L^*$	J_L^*	$\gamma J_L^* - \frac{1}{2}\ln 2$	$\gamma J_L^* - \frac{1}{2}\ln 2$
P_2	J_L^*	$2J_L^*$	J_L^*	$-\gamma J_L^* + \frac{1}{2}\ln 2$	$\gamma J_L^* - \frac{1}{2}\ln 2$
P_3	0	0	$4J_L^*$	0	$2\gamma J_L^* + \ln 2$

Because the exact values of the critical points $(J^*, L^*, K^*, H^*, \Delta^*)$ are known [18, 19], we can get the counterpart $(J_1^*, L_1^*, K_1^*, H_1^*, \Delta_1^*)$. For convenience, we show the results of $(J^*, L^*, K^*, H^*, \Delta^*)$ of the two-dimensional regular $\mathcal{L}$ in table 1. In terms of the first two group results $P_1(J_L^*, -2J_L^*, J_L^*, \gamma J_L^* - \frac{1}{2}\ln 2, \gamma J_L^* - \frac{1}{2}\ln 2)$ and $P_2(J_L^*, 2J_L^*, J_L^*, -\gamma J_L^* + \frac{1}{2}\ln 2, \gamma J_L^* - \frac{1}{2}\ln 2)$, the equations (5) are reduced as

$$\left\{\begin{array}{l} H_1^* = \Delta_1^* \\ J_1^* = -L_1^*/2 = K_1^* \end{array}\right. \tag{6}$$

and

$$\left\{\begin{array}{l} y^2(2+x^2y)^{\gamma} = (2+y)^{\gamma}R_2 \\ y(2+xy)^{\gamma} = (2+y)^{\gamma}R_4 \end{array}\right. \tag{7}$$

for P_1 and

$$\left\{\begin{array}{l} H_1^* = -\Delta_1^* \\ J_1^* = L_1^*/2 = K_1^* \end{array}\right. \tag{8}$$

and

$$\left\{\begin{array}{l} y^2(2+x^2y)^{\gamma} = (2+y)^{\gamma}R_2 \\ y(2+xy)^{\gamma} = (2+y)^{\gamma}R_4 \end{array}\right. \tag{9}$$

for P_2, where $x = e^{4K_1^*}$, $y = e^{-2\Delta_1^*}$, $R_2 = 4$ and $R_4 = 2(e^{2J_L^*})^{-\gamma}$. J_L^* is the critical point of the Ising model on the regular two-dimensional lattice $\mathcal{L}$.

From the third group of results, we get

$$J_1^* = L_1^* = H_1^* = 0 \tag{10}$$

and

$$\left\{\begin{array}{l} y^2(2x^2y+1)^{\gamma} = (2y+1)^{\gamma}R_1 \\ y(2xy+1)^{\gamma} = (2y+1)^{\gamma}R_3 \end{array}\right. \tag{11}$$

where $x = e^{K_1^*}$, $y = e^{-\Delta_1^*}$, $R_1 = \frac{1}{4}$ and $R_3 = \frac{1}{2}(e^{2J_L^*})^{-\gamma}$. Letting $a = y/2$ and $b = x$ in (7) or (9), or $a = 2y$ and $b = x$ in (11), we get a unified form of equations

$$\begin{cases} a^2(ab^2+1)^\gamma = (a+1)^\gamma \\ a(ab+1)^\gamma = \dfrac{(a+1)^\gamma}{z^\gamma} \end{cases} \tag{12}$$

where $z = e^{2J_L^*}$. Eliminating b and setting $a^{\frac{1}{\gamma}} = c$ in (12) leads to

$$\left(\frac{1}{z^2} - 1\right)c^\gamma + c^2 - \frac{2c}{z} + \frac{1}{z^2} = 0. \tag{13}$$

So we can get the exact results of c, and then a and b, and x and y, and finally $(J_1^*, L_1^*, K_1^*, H_1^*, \Delta_1^*)$. The numerical results of (K^*, Δ^*) and (K_1^*, Δ_1^*) for the spin-1 model on initial lattices and the corresponding one-site decorated lattices, respectively, are summarized in table 2. The J_1^*, L_1^* and H_1^* can be calculated through the equations (6), (8) and (10).

Table 2. Summaries of three exact numerical results of (K^*, Δ^*) and (K_1^*, Δ_1^*) for the general spin-1 model on regular lattices and the corresponding one-site decorated lattices, respectively.

		Triangle ($\gamma = 6$)	Square ($\gamma = 4$)	Kagomé ($\gamma = 4$)	Honeycomb ($\gamma = 3$)
P_1, P_2	K_1^*	0.2747	0.4407	0.4666	0.6585
	Δ_1^*	1.3013	1.4162	1.5197	1.6289
P_3	K_1^*	1.0986	1.7627	1.8663	2.6339
	Δ_1^*	3.9890	4.2186	4.4257	4.6440
		Corresponding one-site decorated lattice			
		$\gamma = 3$	$\gamma = 8/3$	$\gamma = 8/3$	$\gamma = 12/5$
P_1, P_2	K_1^*	0.6933	0.8410	0.8751	1.0380
	Δ_1^*	1.8274	1.9283	2.0166	2.1535
P_3	K_1^*	2.7732	3.3639	3.5002	4.1521
	Δ_1^*	5.0411	5.2429	5.4195	5.6932

We notice from table 2 that the critical temperatures for the triangle, square, kagomé and honeycomb lattices, and for their corresponding one-site decorated lattices increase in the same order as those for the Ising model [21]. However, for the Ising model, Syozi [21] proposed that the lattice with heterogeneity in the coordination number has higher critical temperature than the homogeneous lattice with the same coordination number. For example, the critical temperature for the diced lattice is higher than that for the square lattice because both of them have the same average coordination number, but the former has two types of coordination number while the latter has only one type. This view is true of the Ising model on the Asanoha (Hemp-leaf), triangle, diced, square, kagomé, honeycomb and 3–12 lattices, but it is not exactly proven. From table 2, we find the one-site decoration triangle lattice ($\gamma = 3$), which has two kinds of inequivalent lattice point having coordination number six and two in the ratio of 1:3, has lower temperature than the honeycomb lattice ($\gamma = 3$). This result is in disagreement with that for the Ising model. The difference prompts us to reconsider the correctness of Soyio's view. We calculate a series of critical temperatures for the Ising model on the decorated lattice shown in table 3 using the formula [21]

$$e^{2j} = \frac{(e^{2l}+1)^{n+1} + (e^{2l}-1)^{n+1}}{(e^{2l}+1)^{n+1} - (e^{2l}-1)^{n+1}} \tag{14}$$

where $j = \frac{1}{k_B T_c}$ is the critical point for the Ising model on any regular lattice and l is that for the corresponding n-site decorated lattice. We can see from table 3 that the critical temperatures have no exact relations with the average coordination number and the type of coordination number, even for the Ising model.

Table 3. Relations between the critical points of the Ising model for the n-site decorated lattices and their average coordination numbers. h_5 denotes the five-site decorated honeycomb lattice, and so on.

Average of $R = \frac{2(n+1)\gamma}{(2+n\gamma)}$	exp($2j$)	Lattice
2.1176	21.8610	h_5
2.1429	18.2231	h_4
2.1818	14.5867	h_3
2.1818	14.4558	k_5
2.1818	13.6396	s_5
2.2222	12.0550	k_4
2.2222	11.3753	s_4
2.2500	10.9534	h_2
2.2500	9.1485	t_5
2.2857	9.6564	k_3
2.2857	9.1134	s_3
2.3077	7.6371	t_4
2.4000	7.3276	h_1
2.4000	7.2625	k_2
2.4000	6.8565	s_2
2.4000	6.1294	t_3
2.5714	4.6289	t_2
2.6667	4.8800	k_1
2.6667	4.6116	s_1
3.0000	3.1463	t_1

3. Conclusion

We have studied the general spin-1 model on two-dimensional decorated lattices by decoration-iteration transformation. We have found three exact critical points for each case. Because the decorated lattices have two types of coordination number, and various constructions of the decorated lattices can present nearly continuous average coordination number, just like the fractal can show nearly continuous fractal dimension, they are very useful objects in studies of the relation between a certain physical quantity and the coordination number. We find the critical temperatures of both the Ising model and the spin-1 model have no exact relation with the coordination number, not only for the average coordination number, but also for the types of coordination number.

Our method can easily be extended to the studies of the spin-1 model on the two-site, three-site and more-site decorated lattices. We expect the results will provide more evidence to support our view.

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